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Light front quantization by Dirac's method

M Huszár

Central Research Institute for Physics, Budapest 114, PO Box 49, Hungary

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Abstract. Commutation relations between fields and their conjugate momenta cannot be imposed on the light front $x^0+x^3=\text{constant}$ without any further ado because, as a rule, canonical momenta are functions of the fields themselves rather than of the velocities leading to constraints between fields and momenta. To perform light front quantization, Dirac's method for the quantization of constrained systems is applied. Quantization is carried out through the Dirac bracket evaluated by means of constraints. An interesting relation between first-class constraints and boundary conditions is pointed out. Electrodynamics is quantized in a similar manner. If the gauge $A^+=0$ is imposed, constraints become second class, subsidiary conditions do not arise but constraints modify the result of naive quantization in a definite way.

1. Introduction

In the instant form of dynamics (Dirac 1949) the initial conditions are given on the hyperplane $x^0=\text{constant}$ and the Hamiltonian describes the evolution of a system in time. In the light front formulation initial conditions are given on the hyperplane $x^+=x^0+x^3=\text{constant}$ and correspondingly, the commutation relations are prescribed on a plane $x^+=\text{constant}$. In spite of the formal similarity the latter case differs significantly from the instant form. The situation can be simply illustrated in the case of a scalar field. In the instant form the kinetic energy of the Lagrangian contains the term $\frac{1}{2}\partial_0\phi\,\partial_0\phi$, therefore, the canonically conjugate momentum $\pi=\partial\mathscr{L}/\partial\partial_0\phi=\partial_0\phi$ is expressed in terms of the velocity. In the light front formulation, however, the corresponding term is $\partial_+\phi\,\partial_-\phi$ and the conjugate momentum thus takes the form $\pi=\partial\mathscr{L}/\partial\partial_+\phi=\partial_-\phi$ which is not a velocity since it contains a derivative with respect to $x^-=x^0-x^3$, rather than with respect to x^+ which plays the role of time. Thus from the point of view of Hamiltonian mechanics a constraint between condinates and momenta arises resulting in the following questions: what happens to the constraints in the course of time and are they respected by the Hamiltonian? How should the commutation relations be given for a constrained system? The answers to these questions provided by Dirac (1964) may be summarized as follows. One has to work out the conditions whereby the constraints hold in time, i.e. they should be consistent with the Hamiltonian of the system. Then, generally speaking, new constraints arise and the procedure is to be continued until the system of constraints becomes closed. Constraints obtained in this manner are divided into two groups. First-class constraints have to be considered as subsidiary conditions when quantizing the theory, i.e. these constraints restrict the Hilbert space of physical states, whereas second-class constraints are strong operator identities which modify the commutation relations in a definite way. This method has been applied in the quantization of gauge fields (Hasenfratz and Hraskó 1974) as well as in two-dimensional quantum electrodynamics for constructing a bag theory (Shalloway 1975). For other applications of light front quantization the reader is referred to Weinberg (1966), Bjorken (1969), Bjorken and Paschos (1969), Bjorken *et al* (1971), Fubini and Furlan (1965), Drell *et al* (1969a, b, 1970a, b), Drell and Yan (1970a, b) and Yabuki (1975). After circulation of the preprint of the present paper, A J Hanson has drawn my attention to a book by Hanson *et al* (1976), where an approach similar to ours is presented.

In addition to the Hamiltonian method of Dirac the problem of quantization can be approached by Lagrangian theory too. Chang and Ma (1969a, b) and Chang *et al* (1973, see also Rohrlich 1971) used the Lagrangian approach and starting with Schwinger's action principle the generators induced by variation of field on a surface $x^+ = \text{constant}$ are constructed. The commutator of the generators with fields yields the variation of fields and the commutation relations are to be imposed in accordance with this relation. However, the action principle cannot be applied without any further ado because one cannot know *a priori* which field quantities have to be considered independent on the light front (for details, see Chang *et al* 1973, pp 1135, 1144). In Dirac's method no ambiguity arises. In addition to mathematical rigour, this method is preferable in another respect; namely, in Schwinger's method the generators of the Poincaré group are obtained from the action principle and relativistic covariance is proved by showing that these generators satisfy the Poincaré algebra through the field commutators. Nevertheless, it is still to be shown that the field theory obtained in this way is the quantization of just that classical theory we intended to quantize. This problem does not arise at all in Dirac's method since we start with a classical theory where constraints are present and we quantize this theory. On the other hand, some disadvantage results from using Dirac's method here in that Bose and Fermi systems cannot be treated on the same basis. As Fermi systems have no classical limit, the classical field theory to be quantized does not exist. Although, it is possible to introduce classical systems which exhibit a Fermi-like behaviour and Dirac's method can be formally extended to this case, it turns out, however, that the transition from the Poisson bracket to the anticommutator would need to be accomplished by a factor depending on the total differential added to the Lagrangian at the start (Kálnay 1973). This non-physical conclusion means that in quantization of Fermi systems the Schwinger method has to be applied, since here the requirements of the classical limit do not arise.

The paper is organized as follows. In § 2 light front quantization of the real scalar field is performed. It is shown that in addition to second-class constraints a first-class constraint is present which, however, vanishes whenever the field is subject to correct light front boundary conditions.

In § 3 light front quantization of quantum electrodynamics is performed. It is shown that if the gauge $A^+ = 0$ is imposed all constraints become second class. These modify the field commutator in a definite manner and no restriction of the Hilbert space to physical states is necessary.

2. Light front quantized scalar field

The familiar light front variables

$$x^+ = x^0 + x^3, \qquad x^- = x^0 - x^3, \qquad \boldsymbol{x} = (x^1, x^2)$$

will be used. With these the scalar product can be written in the form

$$x_\mu y^\mu = \tfrac{1}{2}(x^+y^- + x^-y^+) - x^k y^k.$$

Covariant and contravariant derivatives are related by

$$\partial_+ = \frac{\partial}{\partial x^+} = \tfrac{1}{2}\partial^-, \qquad \partial_- = \frac{\partial}{\partial x^-} = \tfrac{1}{2}\partial^+, \qquad \partial_k = \frac{\partial}{\partial x^k} = -\partial^k. \tag{2.1}$$

The time variable is now x^+ and x^-, x^1, x^2 are continuous labels for enumerating coordinates. Thus if the scalar field is considered as a mechanical system with generalized coordinates $q_a(t)$, the correspondence $q_a(t) \to \phi_{x^-,x^1,x^2}(x^+) = \phi(x)$ should be made. Denoting by $\pi(x)$ the momentum conjugate to $\phi(x)$, in the continuous case the Poisson bracket (PB) is given by

$$\{F(x), G(y)\}_{x^+=y^+} = \int dz^-\, d^2z \left(\frac{\delta F(x)}{\delta\phi(z)}\frac{\delta G(y)}{\delta\pi(z)} - \frac{\delta F(x)}{\delta\pi(z)}\frac{\delta G(y)}{\delta\phi(z)}\right). \tag{2.2}$$

Here and subsequently the PB is related to equal x^+ values. When multicomponent fields are present summation over discrete variables in addition to integration is understood.

For quantizing the scalar field we start with the action

$$S = \int d^4x \mathscr{L}(x) = \tfrac{1}{2}\int dx^+\, dx^-\, d^2x \mathscr{L}(x) = \int dx^+ L$$

where

$$L = \tfrac{1}{2}\int dx^-\, d^2x \mathscr{L}(x) \tag{2.3}$$

is the Lagrangian, which, for the real scalar field takes the form

$$L = \int dx^-\, d^2x (\partial_+\phi\partial_-\phi - \tfrac{1}{4}\partial_k\phi\partial_k\phi - \tfrac{1}{4}m^2\phi^2). \tag{2.4}$$

The momentum conjugate to ϕ is

$$\pi(x) = \delta L/\delta\partial_+\phi(x) = \partial_-\phi(x).$$

It is evident that we have a singular Lagrangian (Dirac 1964) since the momentum $\partial_-\phi(x)$ is expressed through the coordinates rather than through the velocity. One therefore has the primary constraint

$$\chi = \pi - \partial_-\phi = 0. \tag{2.5}$$

Actually this is not a single constraint but a continuously infinite set of constraints at each point $(x^-, \boldsymbol{x})$. The Hamiltonian is given by

$$H = \int dx^-\, d^2x \pi(x)\partial_+\phi(x) - L = \tfrac{1}{4}\int dx^-\, d^2x (\partial_k\phi\partial_k\phi + m^2\phi^2). \tag{2.6}$$

This does not contain the velocities, nor even the momenta. The constraints are now added by multipliers to the Hamiltonian:

$$H_T = H + \int dx^-\, d^2x\, u(x)(\pi(x) - \partial_-\phi(x)). \tag{2.7}$$

Using equations (2.2) and (2.7) one gets the consistency condition for the constraints:

$$\partial_+\chi = \partial_+(\pi - \partial_-\phi) = \{\chi, H_T\} = \tfrac{1}{2}\partial_k\partial_k\phi - \tfrac{1}{2}m^2\phi - 2\partial_-u.$$

The condition for the constraint to hold in time is

$$\partial_-u = \tfrac{1}{4}(\partial_k\partial_k\phi - m^2\phi). \tag{2.8}$$

The constraint holds in time provided that $u(x)$ satisfies (2.8). No further step is necessary. It is easily seen that an equation for $u(x)$ means at the same time that the constraint is of the second class. From now on $u(x)$ in H_T is considered as satisfying (2.8).

The equation of motion for $\phi(x)$ is simply

$$\partial_+\phi = \{\phi, H_T\} = u(x).$$

Substituting this into the condition (2.8) the familiar equation of motion is obtained:

$$(4\partial_+\partial_- - \partial_k\partial_k + m^2)\phi = (\Box + m^2)\phi = 0.$$

As the constraint is of the second class, no subsidiary condition should be imposed. To perform the quantization one has to evaluate the Dirac bracket according to the definition (Dirac 1964)

$$\{F, G\}^* = \{F, G\} - \sum_{A,B} \{F, \chi_A\} C^{-1}_{AB} \{\chi_B, G\}$$

where F and G are two arbitrary dynamical quantities, χ_A are second-class constraints and C^{-1}_{AB} is the inverse of the matrix of constraints $C_{AB} = \{\chi_A, \chi_B\}$ formed from the second-class constraints. In the present case

$$C_{AB} \to C(x, y) = \{\chi(x), \chi(y)\} = -2\partial_-\delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}). \tag{2.9}$$

(The derivative is always related to the first variable, i.e. $\partial_-\delta(x^- - y^-) = (\partial/\partial x^-)\delta(x^- - y^-)$.)

This matrix can be easily inverted:

$$C^{-1}(x, y) = -\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}).$$

The inverse, however, is not unique. This problem is discussed later on.

Using the ordinary PB of $\phi(x)$ and $\pi(y)$,

$$\{\phi(x), \pi(y)\} = \delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}) \tag{2.10}$$

the Dirac bracket according to (2.9), is

$$\{\phi(x), \pi(y)\}^* = \{\phi(x), \pi(y)\} - \int du^-\, d^2u\, dv^-\, d^2v \{\phi(x), \chi(u)\} C^{-1}(u, v) \{\chi(v), \pi(y)\}$$
$$= \delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}) - \tfrac{1}{2}\delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}) = \tfrac{1}{2}\delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}). \tag{2.11}$$

Transition from the Dirac bracket to the commutator should be performed according to $\{\phi, \pi\}^* \to -\mathrm{i}[\phi, \pi]$, therefore

$$[\phi(x), \pi(y)] = \tfrac{1}{2}\mathrm{i}\delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}), \qquad (x^+ = y^+). \tag{2.12}$$

It should be noted that the naive way of quantization which ignores the constraints gives a false result. Subtraction of the degrees of freedom frozen in modifies the result of naive quantization by a factor of 1/2.

A second-class constraint can be replaced by zero within the Dirac bracket, therefore

$$\{\phi(x), \pi(y)-\partial_-\phi(y)\}^*=0$$

or

$$\{\phi(x), \pi(y)\}^*=\{\phi(x), \partial_-\phi(y)\}^*.$$

With this the quantization condition can be written as

$$[\phi(x), \partial_-\phi(y)]=\tfrac{1}{2}\mathrm{i}\delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}) \qquad (x^+=y^+). \tag{2.13}$$

By evaluating the Dirac bracket of the fields one gets

$$\{\phi(x), \phi(y)\}^*=-\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}) \tag{2.14}$$

therefore,

$$[\phi(x), \phi(y)]=\tfrac{1}{4}\mathrm{i}\epsilon(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y})$$

which is, of course, consistent with (2.13). The commutator of $\pi(x)$, $\pi(y)$ can be obtained in a similar way but it is simpler to take the derivative of equation (2.13) with respect to x^-.

It is worthwhile looking at the origin of non-uniqueness of the inverse of the constraint matrix

$$C(x, y)=\{\chi(x), \chi(y)\}=-2\partial_-\delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}).$$

It will be shown that there is an interesting connection between this ambiguity and the characteristic initial value problem. Namely, it will be shown that:

(*a*) there is a hidden set of first-class constraints in the problem;

(*b*) this is closely related to the characteristic initial value problem and the above non-uniqueness, as well as the first-class constraints, becomes eliminated whenever the field $\phi(x)$ is subject to correct light front initial conditions.

The inverse of $C(x, y)$ is not unique because

$$C'^{-1}=-\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y})-\alpha(x^+, \boldsymbol{x}; y)$$

also satisfies the relation

$$\int \mathrm{d}z^-\, \mathrm{d}^2z\, C(x, z)C'^{-1}(z, y)=\delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}).$$

Here α is an arbitrary function of its arguments.

The matrix of constraints has to be constructed from the second-class constraints. The constraints $\chi(x)$ are apparently of the second class since the matrix $\{\chi(x), \chi(y)\}$ does not vanish identically for any fixed value of x. It is easy to see, however, that there exists a linear combination of constraints commuting with each constraint $\chi(x)$. Namely, the combination

$$\bar{\chi}=\int \mathrm{d}x^-\, \mathrm{d}^2x\, \Lambda(x^+, \boldsymbol{x})\chi(x) \tag{2.15}$$

commutes with each of the remaining constraints, where $\Lambda(x^+, \boldsymbol{x})$ is an arbitrary function independent of x^-:

$$\{\bar{\chi}, \chi(x)\}=0. \tag{2.16}$$

The reason for non-uniqueness of the inverse constraint matrix was simply the presence of the first-class constraints in the set $\chi(x)$. The first-class constraints $\bar{\chi}$ appear in the Hamiltonian as well. To see this recall that the consistency condition for the constraint $\chi(x)$ is

$$\partial_- u = \tfrac{1}{4}(\partial_k \partial_k \phi - m^2 \phi). \tag{2.17}$$

The constraint holds in time provided $u(x)$ is any particular solution of (2.17). The general solution of (2.17) reads

$$\begin{aligned} u(x) &= \tfrac{1}{8} \int \mathrm{d}y^- \, \mathrm{d}^2 y \, \epsilon(x^- - y^-) \delta^2(\boldsymbol{x} - \boldsymbol{y})(\partial_k \partial_k \phi(y) - m^2 \phi(y)) - \Lambda(x^+, \boldsymbol{x}) \\ &= u^0(x) - \Lambda(x^+, \boldsymbol{x}) \end{aligned}$$

where the part of $u(x)$ determined uniquely has been denoted by $u^0(x)$. The total Hamiltonian can be written with this as

$$H_{\mathrm{T}} = H + \int \mathrm{d}x^- \, \mathrm{d}^2 x \, u^0(x) \chi(x) - \int \mathrm{d}x^- \, \mathrm{d}^2 x \, \Lambda(x^+, \boldsymbol{x}) \chi(x).$$

It has been proved by Dirac that the contribution arising from the homogeneous solution of the equation for $u(x)$ is always a first-class constraint. However, without referring to this it can be verified directly that

$$\{\bar{\chi}, \chi(x)\} = \left\{ \int \mathrm{d}y^- \, \mathrm{d}^2 y \, \Lambda(x^+, \boldsymbol{y}) \chi(y), \chi(x) \right\} = 0$$

as was indicated in (2.16).

The appearance of the first-class constraint has an interesting consequence. Physical quantities are those whose PB vanish with each first-class constraint. It follows that in this sense the field $\phi(x)$ itself is not a physical quantity since

$$\{\phi(x), \bar{\chi}\} = \Lambda(x^+, \boldsymbol{x}) \neq 0.$$

Whereas the derivative of the field with respect to x^- is a physical quantity since

$$\{\partial_- \phi, \bar{\chi}\} = 0.$$

The situation is analogous to electrodynamics: the field $\phi(x)$ plays the role of the vector potential and $\partial_- \phi(x)$ is the 'field strength'. The main difference, however, is that in contrast to electrodynamics the gauge freedom vanishes whenever the field $\phi(x)$ is subject to correct boundary conditions. Let $\partial_- \phi(x) = \psi(x)$, then

$$\phi(x) = \int_{-\infty}^{x^-} \mathrm{d}x^- \psi(x) - \Lambda(x^+, \boldsymbol{x}).$$

A possible boundary condition is that $\phi(x)$ vanishes for each x^+ at $x^- \to -\infty$ (Rohrlich 1971). The requirement of this boundary condition for $\phi(x)$ implies $\Lambda(x^+, \boldsymbol{x}) = 0$. It is possible to impose the boundary condition $\phi(x^- = \infty) = -\phi(x^- = -\infty)$. In this case $\Lambda(x^+, \boldsymbol{x})$ is determined uniquely again:

$$\Lambda(x^+, \boldsymbol{x}) = \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x^- \psi(x).$$

It is seen that the gauge freedom is only apparent since the gauge function is determined by the boundary conditions.

Let us proceed to the problem of the Dirac bracket. A linear manifold of constraints $\chi(x)$ is given, $\int dx^- d^2x\, a(x)\chi(x)$, where $a(x)$ is an arbitrary function. It contains a linear subspace of the first-class constraints $\int dx^- d^2x\, \Lambda(x^+, \boldsymbol{x})\chi(x)$. The complementer space gives the second-class constraints needed for the evaluation of the Dirac bracket. Therefore, an arbitrary linear combination of constraints has to be decomposed uniquely into two parts in such a way that one of them should yield the first-class constraints $\int dx^- d^2x\, \Lambda(x^+, \boldsymbol{x})\chi(x)$. One can write

$$\int dx^- d^2x\, a(x)\chi(x) = \int dx^- d^2x(a(x) + \Lambda(x^+, \boldsymbol{x}))\chi(x) - \int dx^- d^2x\, \Lambda(x^+, \boldsymbol{x})\chi(x). \tag{2.18}$$

However, this decomposition is not unique but becomes unique by imposing a definite boundary condition on the coefficient $a_1(x) = a(x) + \Lambda(x^+, \boldsymbol{x})$, e.g.

$$a_1(x^- = \infty) = -a_1(x^- = -\infty). \tag{2.19}$$

In this case $\Lambda(x^+, \boldsymbol{x})$ is expressed in terms of $a(x)$ in a unique way:

$$\Lambda(x^+, \boldsymbol{x}) = \tfrac{1}{2}(a(x^+, x^- = \infty, \boldsymbol{x}) + a(x^+, x^- = -\infty, \boldsymbol{x}).$$

Therefore, second-class constraints are obtained by combining the constraints $\chi(x)$ with coefficients satisfying condition (2.19). This requirement is met for example by the following linear combination:

$$\tilde{\chi}(x) = \tfrac{1}{4}\int dy^- d^2y\, \epsilon(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y})\chi(y).$$

Then the matrix of constraints becomes

$$\bar{C}(x, y) = \{\tilde{\chi}(x), \tilde{\chi}(y)\} = \tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}).$$

It already has a unique inverse,

$$\bar{C}^{-1}(x, y) = 2\partial_-\delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}).$$

The Dirac bracket of $\phi(x)$ and $\pi(y)$ can now be written

$$\{\phi(x), \pi(y)\}^* = \{\phi(x), \pi(y)\} - \int du^- d^2u\, dv^- d^2v\{\phi(x), \tilde{\chi}(u)\}\bar{C}^{-1}(u, v)\{\tilde{\chi}(v), \pi(y)\}$$
$$= \tfrac{1}{2}\delta(x^- - y^-)\delta^2(\boldsymbol{x} - \boldsymbol{y}).$$

Hence the above more vigorous treatment leads to the same result for the Dirac bracket of $\phi(x)$ and $\pi(y)$ as was obtained formerly in (2.11); however, it gives an insight into an interesting connection between boundary conditions and first-class constraints. A similar procedure could be carried out also for the electromagnetic field. However, as the main features of this rigorous treatment can also be seen from the scalar field the discussion of this problem will be omitted.

For the proof of covariance of the theory the generators of the Poincaré group, $M_{\mu\nu}$, P_μ have to be shown to satisfy the Lie algebra of the Poincaré group. The ten generators can be given in the form

$$P_+ = H = \tfrac{1}{4}\int dx^- d^2x(\partial_k\phi\partial_k\phi + m^2\phi^2)$$

$$P_- = \int dx^- d^2x\, \pi\partial_-\phi$$

$$P_k = \int dx^- d^2x\, \pi\partial_k\phi$$

$$M_{-k} = \int dx^- d^2x\, \pi(x_-\partial_k - x_k\partial_-)\phi$$

$$M_{12} = \int dx^- d^2x\, \pi(x_1\partial_2 - x_2\partial_1)\phi$$

$$M_{+-} = \int dx^- d^2x(x_+\pi\partial_-\phi - \tfrac{1}{2}x_-\mathscr{H})$$

$$M_{+k} = \int dx^- d^2x(x_+\pi\partial_k\phi - \tfrac{1}{2}x_k\mathscr{H})$$

where $\mathscr{H}$ is the Hamiltonian density

$$\mathscr{H} = \tfrac{1}{2}(\partial_k\phi\partial_k\phi + m^2\phi^2).$$

It is straightforward but somewhat tedious to show that as a consequence of field commutators the above generators really satisfy the Lie algebra of the Poincaré group.

3. Light front quantum electrodynamics

For quantization of the electromagnetic field first the extended Hamiltonian is evaluated in an arbitrary gauge and then the consistency conditions of the constraints are worked out. In this way also first-class constraints arise which ought to be treated as subsidiary conditions. The gauge $A^+ = 0$ is imposed, however, converting each constraint into a second-class one.

We start with the action integral

$$S = \int d^4x\mathscr{L}(x) = \int dx^+ L(x)$$

where

$$L(x) = \tfrac{1}{2}\int dx^- d^2x\mathscr{L}(x). \tag{3.1}$$

The Lagrangian density of the electromagnetic field can be given as

$$\mathscr{L}(x) = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$

In light front coordinates the components of $F_{\mu\nu}$ are expressed in terms of field strengths by

$$F_{+1} = -\tfrac{1}{2}F^{-1} = -\tfrac{1}{2}(E_1 - B_2), \qquad F_{+2} = -\tfrac{1}{2}F^{-2} = -\tfrac{1}{2}(E_2 + B_1)$$
$$F_{-1} = -\tfrac{1}{2}F^{+1} = -\tfrac{1}{2}(E_1 + B_2), \qquad F_{-2} = -\tfrac{1}{2}F^{+2} = -\tfrac{1}{2}(E_2 - B_1)$$
$$F_{+-} = -\tfrac{1}{4}F^{+-} = \tfrac{1}{2}E_3, \qquad F_{12} = F^{12} = B_3.$$

In terms of these the Lagrangian (3.1) can be written

$$L=\int dx^- d^2x[(F_{+-})^2+F_{+k}F_{-k}-\tfrac{1}{8}F_{ik}F_{ik}].$$

The vector potentials A_μ are considered as dynamical coordinates, their conjugate momenta being

$$\pi^+=\delta L/\delta\partial_+A_+=0 \qquad \text{(constraint)} \tag{3.2}$$

$$\pi^k=\delta L/\delta\partial_+A_k=F_{k-} \qquad \text{(constraint)} \tag{3.3}$$

$$\pi^-=\delta L/\delta\partial_+A_-=2F_{-+}. \tag{3.4}$$

The first three of the canonical momenta π^+, π^1, π^2 yield constraints, whereas the last equation (3.4) does not because π^- is related to the velocity and coordinate, $\pi^-=2(\partial_+A_- - \partial_-A_+)$.

The Hamiltonian has the form

$$H=\int dx^- d^2x\,\pi^\mu\partial_+A_\mu - L$$

$$=\int dx^- d^2x[\tfrac{1}{4}(\pi^-)^2-(\partial_-\pi^-+\partial_k\pi^k)A_+ +\tfrac{1}{8}F_{ik}F_{ik}]. \tag{3.5}$$

The total Hamiltonian is obtained by adding the constraints to (3.5) by Lagrangian multipliers:

$$H_T=H+\int dx^- d^2x[u_+\pi^+ + u_k(\pi^k+F_{-k})]. \tag{3.6}$$

The consistency condition for the constraint $\pi^+=0$ is

$$\partial_+\pi^+=\{\pi^+,H_T\}=\partial_-\pi^-+\partial_k\pi^k.$$

A new, secondary constraint, corresponding to case (*b*) of § 2 arises, therefore we have

$$\partial_-\pi^-+\partial_k\pi^k=0. \tag{3.7}$$

The secondary constraint must hold in time, again:

$$\partial_+(\partial_-\pi^-+\partial_k\pi^k)=\{\partial_-\pi^-+\partial_k\pi^k, H_T\}=0.$$

This condition does not lead to a new constraint; instead it turns to an identity.

The condition for the remaining constraints (3.3) takes the form

$$\partial_+(\pi^k+F_{-k})=\{\pi^k+F_{-k}, H_T\}=\tfrac{1}{2}\partial_iF_{ik}-\tfrac{1}{2}\partial_k\pi^-+2\partial_-u_k=0$$

i.e. functions u_k have to satisfy

$$\partial_-u_k=\tfrac{1}{4}(\partial_k\pi^-+\partial_iF_{ki}). \tag{3.8}$$

One now has to decide whether the constraints are of the first class. If they are one must add them to the Hamiltonian with some multipliers. The secondary constraint (3.7) proves to be of the first class because

$$\{\partial_-\pi^-(x)+\partial_k\pi^k(x), \partial_-\pi^-(y)+\partial_k\pi^k(y)\}=0$$

$$\{\partial_-\pi^-(x)+\partial_k\pi^k(x), \pi^+(y)\}=0, \qquad \{\partial_-\pi^-(x)+\partial_k\pi^k(x), \pi^l(y)+F_{-l}(y)\}=0.$$

The most general motion of the electromagnetic field is described by the extended Hamiltonian

$$H_E=H_T+\int dx^- d^2v(x)(\partial_-\pi^-+\partial_k\pi^k). \tag{3.9}$$

As the original Hamiltonian (3.5) also contains a term $-(\partial_-\pi^-+\partial_k\pi^k)A_+$ one can incorporate it into the multiplier v by introducing $v'=v-A_+$. Collecting the terms arising from (3.5), (3.6) and (3.9) we finally arrive at the following extended Hamiltonian:

$$H_E=\int dx^- d^2x[\tfrac{1}{4}(\pi^-)^2+\tfrac{1}{8}F_{lk}F_{lk}]+\int dx^- d^2x[u_+\pi^++u_k(\pi^k+F_{-k})+v'(\partial_-\pi^-+\partial_k\pi^k)]. \tag{3.10}$$

The multipliers u_k are considered as satisfying the consistency condition (3.8). The remaining coefficients u_+, v' are completely arbitrary, describing the gauge freedom of the system and the freedom incorporated in them does not concern the time evolution of physical quantities; nevertheless, the coefficients generally show up in equations of motion of vector potentials. It proves useful to impose the gauge $A^+=2A_-=0$, and $\partial_\mu A^\mu=0$ which will restrict the coefficients u_+, v'.

Before proceeding further let us look at the degrees of freedom which describe the electromagnetic field in this gauge. Maxwell equations $\partial_\nu F^{\mu\nu}=0$ reduce to

$$\Box A_\mu=0 \tag{3.11}$$

provided that $\partial_\mu A^\mu=0$. One can still perform a gauge transformation with a gauge function satisfying $\Box\Lambda=0$. This can be used for imposing the gauge $A_-=0$ with which the Lorentz condition assumes the form

$$\partial_\mu A^\mu=2\partial_+A_-+2\partial_-A_+-\partial_kA_k=0 \tag{3.12}$$

i.e.

$$2\partial_-A_+-\partial_kA_k=0. \tag{3.13}$$

This is clearly a constraint for A_+ rather than an equation of motion. The time evolution of A_+ is determined by A_k and thus an equation of motion is obtained merely for two quantities A_1 and A_2.

Let us go back to the extended Hamiltonian (3.10) and consider the consistency condition for the gauge $A_-=0$,

$$\partial_+A_-=\{A_-, H_E\}=\tfrac{1}{2}\pi^--\partial_-v'$$

i.e. the gauge chosen holds in time provided that

$$\partial_-v'=\tfrac{1}{2}\pi^-. \tag{3.14}$$

According to (3.13) the Lorentz condition now reads $2\partial_-A_+-\partial_kA_k=0$. This is a counterpart to the radiation gauge $\operatorname{div}\boldsymbol{A}=0$ imposed in the instant form of electrodynamics. The consistency condition $\partial_+(2\partial_-A_+-\partial_kA_k)=0$ implies the relation between the multipliers

$$2\partial_- u_+ - \partial_k u_k + \partial_k \partial_k v' = 0. \tag{3.15}$$

Hence, the Lorentz condition and the gauge $A^+ = 2A_- = 0$ hold in time if u_+ and v' satisfy relations (3.14) and (3.15).

To sum up, the following constraints have been obtained:

$$\begin{aligned} &\chi_1 = \pi^+ = 0, && \chi_2 = 2\partial_- A_+ - \partial_k A_k = 0, && \chi_3 = A_- = 0 \\ &\chi_4 = \pi^1 + F_{-1} = 0, && \chi_5 = \pi^2 + F_{-2} = 0, && \chi_6 = \partial_- \pi^- + \partial_k \pi^k = 0. \end{aligned} \tag{3.16}$$

The new constraints arising from the choice of gauge are obviously second class for their consistency conditions lead to equations for the multipliers. Indeed, all six types of constraints are of the second class since none of them has zero PB with the remaining ones. The importance of the order of the steps taken should be stressed. Before fixing the gauge the constraint $\chi_6 = \partial_- \pi^- + \partial_k \pi^k$ proved to be a first-class one and was the reason for adding it to H_T in order to obtain the most general motion. At this point the procedure could be finished; in this general formulation we are not bound to choose any particular gauge. However, having fixed the gauge the constraint $\chi_3 = A_-$ fails to commute with χ_6, and consequently, no longer remains a first-class constraint. Each constraint we have is a second-class one, therefore the entire Hilbert space will be physical and no subsidiary condition will need to be taken into account. It was for this reason that the above gauge was chosen.

Before proceeding to the quantization it is worthwhile looking at the equations of motion arising from (3.10), i.e.

$$\begin{aligned} &\partial_+ A_+ = 0, && \partial_+ A_- = 0, && \partial_+ A_k = u_k - \partial_k v' \\ &\partial_+ \pi^+ = 0, && \partial_+ \pi^- = \partial_k u_k, && \partial_+ \pi^k = \tfrac{1}{2}\partial_l F_{kl} - \partial_- u_k. \end{aligned}$$

In addition we have the constraints $\chi_A = 0$ $(A = 1, \ldots, 6)$ and the equations for the multipliers (3.8), (3.14) and (3.15) resulting in

$$(4\partial_+\partial_- - \partial_k \partial_k) A_i = \Box A_i = 0 \qquad (i, k = 1, 2).$$

The potential A_+ is determined by the constraint $\chi_2 = 0$.

The non-vanishing PB's of constraints are

$$\begin{aligned} &\{\chi_1(x), \chi_2(y)\} = 2\mathrm{D}_-(x-y), && \{\chi_2(x), \chi_4(y)\} = -\mathrm{D}_1(x-y) \\ &\{\chi_2(x), \chi_5(y)\} = -\mathrm{D}_2(x-y), && \{\chi_2(x), \chi_6(y)\} = \mathrm{D}_{kk}(x-y) \\ &\{\chi_3(x), \chi_6(y)\} = -\mathrm{D}_-(x-y), && \{\chi_4(x), \chi_4(y)\} = -2\mathrm{D}_-(x-y) \\ &\{\chi_5(x), \chi_5(y)\} = -2\mathrm{D}_-(x-y) \end{aligned} \tag{3.17}$$

where the following notation is used

$$\begin{aligned} &\mathrm{D}_-(x-y) = \frac{\partial}{\partial x^-}\delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}), \qquad \mathrm{D}_k(x-y) = \delta(x^- - y^-)\frac{\partial}{\partial x^k}\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\mathrm{D}_{kk}(x-y) = \delta(x^- - y^-)\left(\frac{\partial}{(\partial x^1)^2} + \frac{\partial}{(\partial x^2)^2}\right)\delta^2(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{3.18}$$

In terms of these, the matrix of constraints is

$$C_{AB}(x,y)=\{\chi_A(x),\chi_B(y)\}=\begin{bmatrix} 0 & 2\mathrm{D}_- & 0 & 0 & 0 & 0 \\ 2\mathrm{D}_- & 0 & 0 & -\mathrm{D}_1 & -\mathrm{D}_2 & \mathrm{D}_{kk} \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{D}_- \\ 0 & -\mathrm{D}_1 & 0 & -2\mathrm{D}_- & 0 & 0 \\ 0 & -\mathrm{D}_2 & 0 & 0 & -2\mathrm{D}_- & 0 \\ 0 & \mathrm{D}_{kk} & -\mathrm{D}_- & 0 & 0 & 0 \end{bmatrix}. \tag{3.19}$$

Each element of this matrix is a function of $(x-y)$ thus denoting a continuously infinite matrix. For the Dirac bracket one has to invert $C_{AB}(x, y)$:

$$C^{-1}_{AB}(x,y)=\begin{bmatrix} -(\mathrm{D}_1^2+\mathrm{D}_2^2)/8\mathrm{D}_-^3 & 1/2\mathrm{D}_- & \mathrm{D}_{kk}/2\mathrm{D}_-^2 & -\mathrm{D}_1/4\mathrm{D}_-^2 & -\mathrm{D}_2/4\mathrm{D}_-^2 & 0 \\ 1/2\mathrm{D}_- & 0 & 0 & 0 & 0 & 0 \\ \mathrm{D}_{kk}/2\mathrm{D}_-^2 & 0 & 0 & 0 & 0 & -1/\mathrm{D}_- \\ -\mathrm{D}_1/4\mathrm{D}_-^2 & 0 & 0 & -1/2\mathrm{D}_- & 0 & 0 \\ -\mathrm{D}_2/4\mathrm{D}_-^2 & 0 & 0 & 0 & -1/2\mathrm{D}_- & 0 \\ 0 & 0 & -1/\mathrm{D}_- & 0 & 0 & 0 \end{bmatrix}. \tag{3.20}$$

This matrix is constructed to satisfy

$$\int \mathrm{d}u^-\,\mathrm{d}^2u\,C_{AB}(x,u)C^{-1}_{BC}(u,y)=\delta_{AC}\delta(x^--y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}).$$

In (3.20) $1/\mathrm{D}_-$ denotes the inverse to $\mathrm{D}_-(x-y)$. It is easily seen that $(1/\mathrm{D})(x,y)=\frac{1}{2}\epsilon(x^--y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y})$. Similarly, $\mathrm{D}_1/4\mathrm{D}_-^2$ is shorthand for the product

$$\frac{\mathrm{D}_1}{4\mathrm{D}_-^2}(x,y)=\tfrac{1}{4}\int \mathrm{d}u^-\,\mathrm{d}^2u\,\mathrm{D}_1(x-u)\mathrm{D}_-^{-2}(u-y).$$

It is easy to verify the following equalities:

$$\left(\frac{1}{\mathrm{D}_-^2}\right)(x,y)=\tfrac{1}{4}\int \mathrm{d}u^-\,\mathrm{d}^2u\,\mathrm{D}^{-1}(x,u)\mathrm{D}^{-1}(u,y)=\tfrac{1}{2}|x^--y^-|\delta^2(\boldsymbol{x}-\boldsymbol{y}) \tag{3.21}$$

$$\left(\frac{\mathrm{D}_k}{4\mathrm{D}_-^2}\right)(x,y)=\tfrac{1}{4}\int \mathrm{d}u^-\,\mathrm{d}^2u\,\mathrm{D}_k(x-u)\left(\frac{1}{\mathrm{D}_-^2}\right)(u,y)=\tfrac{1}{8}|x^--y^-|\frac{\partial}{\partial x^k}\delta^2(\boldsymbol{x}-\boldsymbol{y}) \tag{3.22}$$

$$\left(\frac{\mathrm{D}_{kk}}{4\mathrm{D}_-^2}\right)(x,y)=\tfrac{1}{4}\int \mathrm{d}u^-\,\mathrm{d}^2u\,\mathrm{D}_{kk}(x-u)\left(\frac{1}{\mathrm{D}_-^2}\right)(u,y)=\tfrac{1}{8}|x^--y^-|\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^k}\delta^2(\boldsymbol{x}-\boldsymbol{y}). \tag{3.23}$$

The matrix $(\mathrm{D}_-)^{-3}$ occurring in the 1,1 element of the matrix C^{-1}_{AB} is not well defined. This term appears in the Dirac bracket $\{A_+(x), A_+(y)\}$ only. Here we evaluate, instead, $\{A_+(x), \partial_-A_+(y)\}^*$ which is already defined. This is obviously not a serious problem in that A_+ is not an independent degree of freedom but subject to the constraint $\chi_2=2\partial_-A_+-\partial_kA_k=0$ and the above Dirac bracket merely serves to ensure the validity of this constraint which contains only the combination ∂_-A_+.

The Dirac bracket of quantities F, G is

$$\{F(x), G(y)\}^* = \{F(x), G(y)\} - \int du^- \, d^2u \, dv^- \, d^2v \{F(x), \chi_A(u)\} \times C^{-1}_{AB}(u, v)\{\chi_B(v), G(y)\} \qquad (A, B = 1, 2, \ldots 6). \tag{3.24}$$

For evaluating the Dirac bracket of A_μ and π^ν we need the PB's of these quantities with the constraints:

$$\{A_\mu(x), \chi_A(y)\} = \{\delta^+_\mu D(x-y), 0, 0, \delta^1_\mu D(x-y), \delta^2_\mu D(x-y), -\delta^-_\mu D_-(x-y), -\delta^k_\mu D_k(x-y)\} \tag{3.25}$$

$$\{\chi_A(x), \pi^\nu(y)\} = (0, 2\delta^\nu_+ D_-(x-y) - \delta^\nu_k D_k(x-y), \delta^\nu_- D(x-y), \delta^\nu_- D_1(x-y) - \delta^\nu_1 D_-(x-y), \delta^\nu_- D_2(x-y) - \delta^\nu_2 D_-(x-y), 0). \tag{3.26}$$

In addition to the notation given by (3.18) the notation $D(x-y) = \delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y})$ has been used here.

Substituting the PB's given by (3.25) and (3.26) into the Dirac bracket (3.24) one gets

$$\begin{aligned} &\{A_+(x), \partial_- A_+(y)\}^* = -\tfrac{1}{16}|x^- - y^-|\partial_k\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{A_+(x) A_k(y)\}^* = -\tfrac{1}{8}|x^- - y^-|\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{A_-(x), A_\mu(y)\}^* = 0 \\ &\{A_k(x), A_l(y)\}^* = -\tfrac{1}{4}\delta^l_k\epsilon(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}) \qquad (x^+ = y^+). \end{aligned} \tag{3.27}$$

where $k, l = 1, 2$; $\mu = +, -, 1, 2$ and the derivatives of δ functions are always related to the first variable.

We remind the reader that all PB's between A_μ vanish, therefore, non-zero values on the right-hand side of (3.27) are consequences of the constraints.

Taking into account the PB's

$$\{A_\mu(x), \pi^\nu(y)\} = \delta^\nu_\mu\delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}), \qquad \{\pi^\mu(x), \pi^\nu(y)\} = 0$$

the Dirac brackets between A_μ, π^μ and π^μ, π^ν can be obtained in a similar manner, namely:

$$\begin{aligned} &\{A_\mu(x), \pi^+(y)\}^* = 0 \\ &\{A_-(x), \pi^\mu(y)\}^* = 0 \\ &\{A_k(x), \pi^-(y)\}^* = -\tfrac{1}{4}\epsilon(x^- - y^-)\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{A_+(x), \pi^-(y)\}^* = -\tfrac{1}{8}|x^- - y^-|\partial_k\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{A_+(x), \pi^k(y)\}^* = \tfrac{1}{8}\epsilon(x^- - y^-)\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{A_k(x), \pi^l(y)\}^* = \tfrac{1}{2}\delta^l_k(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{\pi^+(x), \pi^\mu(y)\}^* = 0 \\ &\{\pi^-(x), \pi^-(y)\}^* = \tfrac{1}{4}\epsilon(x^- - y^-)\partial_k\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{\pi^k(x), \pi^-(y)\}^* = -\tfrac{1}{2}\delta(x^- - y^-)\partial_k\delta^2(\boldsymbol{x}-\boldsymbol{y}) \\ &\{\pi^k(x), \pi^l(y)\}^* = \tfrac{1}{2}\delta^l_k\partial_-\delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}) \qquad (x^+ = y^+). \end{aligned} \tag{3.28}$$

Transition to quantization must be accomplished by replacing the Dirac bracket by the commutator according to $\{F, G\}^* \to -\mathrm{i}[\hat{F}, \hat{G}]_{x^+=y^+}$. Thus for instance, the last relation in (3.27) should be replaced by the commutator

$$[A_k(x), A_l(y)] = -\tfrac{1}{4}\mathrm{i}\delta^l_k \epsilon(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y}). \tag{3.29}$$

Since the constraints are of the second class they must hold as operator identities. Thus e.g. the commutator of the constraint $A_- = 0$ with any quantity, must vanish including π^- in spite of the fact that the PB of these quantities is non-zero, $\{A_-(x), \pi^-(y)\} = \delta(x^- - y^-)\delta^2(\boldsymbol{x}-\boldsymbol{y})$. The contribution from the constraints just cancels the δ function in the Dirac bracket. It is easy to check that the remaining constraints are also fulfilled strongly.

The ten generators of the Poincaré group are as follows:

$$P_+ = H = H_{\mathrm{E}} = \int \mathrm{d}x^-\, \mathrm{d}^2x [\tfrac{1}{4}(\pi^-)^2 + \tfrac{1}{8}F^2_{ik}]$$

$$P_- = \int \mathrm{d}x^-\, \mathrm{d}^2x\, \pi^k \partial_- A_k$$

$$P_k = \tfrac{1}{2}\int \mathrm{d}x^-\, \mathrm{d}^2x\, \pi^l \partial_k A_l$$

$$M_{-k} = \int \mathrm{d}x^-\, \mathrm{d}^2x\, \pi^l (\tfrac{1}{2}x_- \partial_k - x_k \partial_-) A_l$$

$$M_{12} = \tfrac{1}{2}\int \mathrm{d}x^-\, \mathrm{d}^2x\, \pi^l (x_1\partial_2 - x_2\partial_1) A_l$$

$$M_{+-} = \int \mathrm{d}x^-\, \mathrm{d}^2x (x_+ \pi^l \partial_- A_l - \tfrac{1}{2}x_- \mathscr{H})$$

$$M_{+k} = \tfrac{1}{2}\int \mathrm{d}x^-\, \mathrm{d}^2x (x_+ \pi^l \partial_k A_l - x_k \mathscr{H})$$

with

$$\mathscr{H} = \tfrac{1}{2}(\pi^-)^2 + \tfrac{1}{4}F_{ik}F_{ik}.$$

For the proof of relativistic covariance it is not sufficient to show that these generators satisfy the Lie algebra of the Poincaré group since we now have a non-covariant gauge $A_- = 0$. If this gauge is imposed in a certain frame of reference and a Lorentz transformation is performed then the condition $A_- = 0$ in general is violated. Therefore, one has also to make a gauge transformation in order to ensure the fulfilment of $A_- = 0$ in the new frame. These combined Lorentz and gauge transformations yield a representation of the Lorentz group. However, the Dirac bracket generates both of these transformations automatically. Suppose an infinitesimal Lorentz transformation of the vector potential is performed. Then the variation of A_- is given by the Dirac bracket of the above generators $M_{\mu\nu}$ with A_-. We know, however, that the second-class constraint $A_- = 0$ can be set equal to zero within the Dirac bracket, i.e. A_- remains unchanged. In other words the Dirac bracket is constructed in such a way that the non-covariant gauge $A_- = 0$ holds in any frame of reference.

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